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SMALL-OBSTACLE EXPANSION IN 3-D INVERSE SCATTERING

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Abstract. This communication concerns an extension of the topological derivative concept, whereby a given cost function (here a misfit function for inverse scattering purposes) is expanded in terms of powers of the characteristic size ε of a small rigid scatterer. This allows to find an estimate of the unknown scatterer on the basis of a minimization of the approximate cost function constructed with the help of the expansion (polynomial in ε). The minimization can be performed pointwise on a search grid and at a relatively modest computational cost, of the order of one forward acoustic solution, whereas a full-fledged (and possibly blind) minimization algorithm needs at the very least one forward solution per iteration. The proposed approach is demonstrated on a numerical example.

1. INTRODUCTION AND BACKGROUND

This study is set in the general framework of inverse scattering of scalar (e.g. acoustic) waves. To identify hidden obstacles from external measurements (e.g. overspecified boundary data) associated with the scattering of known incident waves by the unknown object(s), it is customary to invoke iterative algorithms such as gradient-based optimization techniques. The numerical solution of the forward scattering problem for an assumed obstacle configuration is often a computationally demanding task. Besides, iterative inversion algorithms are sensitive to the choice of initial "guess" (number of components, initial location, shape and size of obstacle(s)).

This has prompted the definition of preliminary probing techniques, which aim at delineating in a computationally fast way the hidden obstacle(s). Such techniques have been investigated based on either the linear sampling [3], an avenue not pursued here, or the topological derivative [2,4,5,8], to which this study is connected.

Consider a reference medium Ω (wavenumber k, mass density ρ) housing an unknown rigid inclusion B^{true} of boundary Γ^{true}. The total field u^{true} (e.g. the acoustic pressure) is governed by problem $\mathcal{P}_k(\Gamma^{\text{true}})$, where $\mathcal{P}_k(\Gamma)$ denotes the generic scattering problem defined by the set of equations

$$\mathcal{P}_k(\Gamma): \qquad \Gamma_\varepsilon(\Delta+\gamma^2k^2)v=0 \text{ (in } \Omega\setminus B) ; \qquad v_{,n}=p \text{ (on } S_{\text{N}}), \quad v=0 \text{ (on } S_{\text{D}}), \quad v_{,n}=0 \text{ (on } \Gamma) \tag{1}$$

where $\boldsymbol{n}$ is the normal on S (outward to Ω) or Γ, and can also be formulated in the form of a generalization of the Lippman-Schwinger integral equation [7]. Consider cost functions of the form

$$\mathcal{J}(\Gamma^\star)=\int_{S^{\text{obs}}}\varphi(u^c,\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi \tag{2}$$

where $v=u^c$ solves problem $\mathcal{P}_k(\Gamma^\star)$, equation (1), for some trial obstacle $\Gamma^\star$. In particular, for the identification of an unknown rigid inclusion B^{true}, the usual least-squares misfit-to-data cost function is defined by (2) with $2\varphi(w,\boldsymbol{\xi})=\overline{(w-u^{\text{true}})}(w-u^{\text{true}})$.

Let $B_\varepsilon(\boldsymbol{x}^\circ)=\boldsymbol{x}^\circ+\varepsilon\mathscr{B}$, where $\mathscr{B}\subset\mathbb{R}^3$ is a fixed bounded open set with boundary $\mathscr{S}$ and volume $|\mathscr{B}|$ containing the origin, define the region of space occupied by an inclusion of (small) size $\varepsilon>0$ containing a fixed sampling point $\boldsymbol{x}^\circ$, with boundary $\Gamma_\varepsilon(\boldsymbol{x}^\circ)$. Denoting by u^ε the solution to the forward problem (1) with Γ replaced by Γ_ε, let $J(\varepsilon,\boldsymbol{x}^\circ)$ denote the value achieved by the cost function (2) when $\Gamma^\star$ is chosen as the boundary of the infinitesimal obstacle $B_\varepsilon(\boldsymbol{x}^\circ)$, i.e.:

$$J(\varepsilon,\boldsymbol{x}^\circ)=\mathcal{J}(\Gamma_\varepsilon(\boldsymbol{x}^\circ))=\int_{S^{\text{obs}}}\varphi(u^\varepsilon,\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi \tag{3}$$

(note that the value $J(0,\boldsymbol{x}^\circ)$ of J for the obstacle-free reference medium does not actually depend on $\boldsymbol{x}^\circ$). One then finds

$$\mathcal{J}(\varepsilon,\boldsymbol{x}^\circ)=\mathcal{J}(0,\boldsymbol{x}^\circ)+\varepsilon^3\,|\mathscr{B}|\,\mathcal{T}(\boldsymbol{x}^\circ)+o(\varepsilon^3) \tag{4}$$

where the *topological derivative* $\mathcal{T}(\boldsymbol{x}^\circ)$ is given by [6]

$$\mathcal{T}(\boldsymbol{x}^\circ)=\text{Re}\big\{\nabla\hat{u}.\boldsymbol{A}(\mathscr{S}).\nabla u-k^2\hat{u}u\big\}(\boldsymbol{x}^\circ) \tag{5}$$

in which u, the acoustic free-field, is defined as the response of the void-free (reference) solid Ω due to given excitation g and hence governed by

$$(\Delta+k^2)u=0\ \ (\text{in }\Omega), \qquad u_{,n}=g\ \ (\text{on } S_{\text{N}}), \quad u=0\ \ (\text{on } S_{\text{D}}), \tag{6}$$

while the adjoint field $\hat{u}$ solves

$$(\Delta + k^2)\hat{u} = 0 \text{ (in } \Omega) \,; \qquad \hat{u}_{,n} = \frac{\partial \varphi}{\partial u} \text{ (on } S^{\text{obs}}), \quad \hat{u}_{,n} = 0 \text{ (on } S_{\text{N}} \setminus S^{\text{obs}}) \quad \hat{u} = 0 \text{ (on } S_{\text{D}}) \tag{7}$$

with the convention

$$\frac{\partial \varphi}{\partial w} \equiv \frac{\partial \varphi}{\partial w_{\text{R}}} - \mathrm{i}\frac{\partial \varphi}{\partial w_{\text{I}}} \qquad (w_{\text{R}} = \text{Re}(w)\,,\ w_{\text{I}} = \text{Im}(w)) \tag{8}$$

and the second-order tensor $\boldsymbol{A}(\mathscr{S})$ has been established for any inclusion shape $\mathscr{S}$ [6]. For the simplest case where $\mathscr{B}$ is the unit sphere, one has

$$A_{ij}(\mathscr{S}) = (3/2)\delta_{ij} \tag{9}$$

2. SMALL-OBSTACLE EXPANSION

In this communication, an extension of the topological derivative is presented, whereby $J(\varepsilon, \boldsymbol{x}^{\text{o}})$ is expanded further in powers of ε. Specifically, the expansion to order $O(\varepsilon^6)$ for 3D acoustic scattering by a hard obstacle of size ε is considered. The order $O(\varepsilon^6)$ is important for cost functions $J(\varepsilon, \boldsymbol{x}^{\text{o}})$ of least-squares format because the perturbations of the residuals featured in $J(\varepsilon, \boldsymbol{x}^{\text{o}})$ are of order $O(\varepsilon^3)$ under the present conditions.

2.1. Expansion of the cost function

To evaluate $J(\varepsilon, \boldsymbol{x}^{\text{o}})$, it is convenient to decompose the total field u^ε as

$$u^\varepsilon = u + v^\varepsilon \tag{10}$$

where u, the free field, is governed by (6) and v^ε, the *scattered field*, is defined on $\Omega_\varepsilon \equiv \Omega \setminus B_\varepsilon$ and solves

$$(\Delta + k^2)v^\varepsilon = 0 \ \text{ (in } \Omega_\varepsilon), \qquad v^\varepsilon_{,n} = 0 \ \text{ (on } S_{\text{N}}), \quad v^\varepsilon = 0 \ \text{ (on } S_{\text{D}}), \quad v^\varepsilon_{,n} = -u_{,n} \ \text{ (on } \Gamma_\varepsilon) \tag{11}$$

For infinitesimal ε the scattered field is expected to vanish, i.e. $\lim_{\varepsilon \to 0} |v^\varepsilon(\boldsymbol{x})| = 0$ $(\boldsymbol{x} \in \Omega_\varepsilon)$, whereas the free-field, by its definition (6), does not depend on ε. One may then expand $\mathcal{J}(\Gamma_\varepsilon)$ with respect to v^ε as

$$J(\varepsilon, \boldsymbol{x}^{\text{o}}) = J(0) + \int_{S^{\text{obs}}} \Big\{ \frac{\partial \varphi}{\partial u}\, v^\varepsilon + \frac{1}{2}\frac{\partial^2 \varphi}{\partial u^2}\, (v^\varepsilon)^2 \Big\}\, \mathrm{d}\Gamma + o(|v^\varepsilon|^2_{S^{\text{obs}}}) \tag{12}$$

Note in particular that the quadratic expansion in v^ε appearing in (12) is exact for a least-squares functional. The first integral term in the r.h.s. can be converted into an integral over the vanishing cavity surface by means of the reciprocity identity

$$\int_{S^{\text{obs}}} \frac{\partial \varphi}{\partial u}\, v^\varepsilon \,\mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} v^\varepsilon \hat{u}_{,n} \,\mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} \hat{u} u_{,n} \,\mathrm{d}\Gamma = 0 \tag{13}$$

where the adjoint state $\hat{u}$ is again defined by (7).

Besides, since both u and $\hat{u}$ are defined inside B_ε, the last integral in (13) can be converted into a domain integral over B_ε by means of the divergence formula. Expansion (12) then takes the form

$$J(\varepsilon, \boldsymbol{x}^{\text{o}}) = \mathcal{J}(\Omega) + \frac{1}{2}\int_{S^{\text{obs}}} \frac{\partial^2 \varphi}{\partial u^2}\, (v^\varepsilon)^2 \,\mathrm{d}\Gamma - \int_{\Gamma_\varepsilon} v^\varepsilon \hat{u}_{,n} \,\mathrm{d}\Gamma + \int_{B_\varepsilon} \big[\nabla u.\nabla \hat{u} - k^2 u\hat{u}\big]\,\mathrm{d}V + o(|v^\varepsilon|^2_{S^{\text{obs}}}) \tag{14}$$

This formulation will be used as a basis for obtaining the sought expansion of $J(\varepsilon, \boldsymbol{x}^{\text{o}})$ in powers of ε. Since the first two integrals involve the traces of the scattered field v^ε on S^{obs} and Γ_ε, respectively, it is necessary to first derive expansions of these traces in powers of ε.

2.2. Expansion of the scattered field

The scattered field v^ε is governed by the integral equation:

$$\frac{1}{2}v^\varepsilon(\boldsymbol{x}) + \int_{\Gamma_\varepsilon} H(\boldsymbol{x}, \boldsymbol{\xi}) v^\varepsilon(\boldsymbol{\xi}) \,\mathrm{d}\Gamma_\xi + \int_{S_{\text{N}}} H(\boldsymbol{x}, \boldsymbol{\xi}) v^\varepsilon(\boldsymbol{\xi}) \,\mathrm{d}\Gamma_\xi - \int_{S_{\text{D}}} G(\boldsymbol{x}, \boldsymbol{\xi}) v^\varepsilon_{,n}(\boldsymbol{\xi}) \,\mathrm{d}\Gamma_\xi$$
$$= \int_{B_\varepsilon} \big[\nabla G(\boldsymbol{x}, \boldsymbol{\xi}).\nabla u(\boldsymbol{\xi}) - k^2 G(\boldsymbol{x}, \boldsymbol{\xi}) u(\boldsymbol{\xi})\big]\,\mathrm{d}V_\xi \qquad (\boldsymbol{\xi} \in S \cup \Gamma_\varepsilon) \tag{15}$$

where the nabla symbol ∇ always indicates the gradient with respect to the integration point $\boldsymbol{\xi}$, and the fundamental kernels $G(\boldsymbol{x}, \boldsymbol{\xi})$ and $H(\boldsymbol{x}, \boldsymbol{\xi})$ are defined by

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{4\pi r} e^{\mathrm{i}kr}\,, \quad H(\boldsymbol{x}, \boldsymbol{\xi}) = \boldsymbol{n}(\boldsymbol{\xi}).\nabla G(\boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{r}.\boldsymbol{n})\frac{\mathrm{i}kr - 1}{4\pi r^3} e^{\mathrm{i}kr} \tag{16}$$

with $\boldsymbol{r} = \boldsymbol{\xi} - \boldsymbol{x}$ and $r = |\boldsymbol{\xi} - \boldsymbol{x}| = |\boldsymbol{r}|$. The right-hand side in (15) stems from an application of the divergence formula to the integral of $-G(\boldsymbol{x}, \boldsymbol{\xi})[\boldsymbol{n}.\nabla u](\boldsymbol{\xi})$ over Γ_ε which is featured in the standard form of that identity, noting that the free-field is defined inside the obstacle and that the unit normal $\boldsymbol{n}$ points to the *interior* of B_ε. In addition, the following integral representation of $v^\varepsilon(\boldsymbol{x})$ holds for any $\boldsymbol{x} \in \Omega_\varepsilon$:

$$v^\varepsilon(\boldsymbol{x}) = -\int_{\Gamma_\varepsilon} H(\boldsymbol{x},\boldsymbol{\xi})v^\varepsilon(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi - \int_{S_\mathrm{N}} H(\boldsymbol{x},\boldsymbol{\xi})v^\varepsilon(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi + \int_{S_\mathrm{D}} G(\boldsymbol{x},\boldsymbol{\xi})v^\varepsilon_{,n}(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi \\ + \int_{B_\varepsilon} \big[\nabla G(\boldsymbol{x},\boldsymbol{\xi})\nabla u(\boldsymbol{\xi}) - k^2 G(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{\xi})\big]\,\mathrm{d}V_\xi \qquad (\boldsymbol{x} \in \Omega_\varepsilon) \quad (17)$$

Determination of the leading asymptotic contribution. The traces on Γ_ε and S of the scattered field are tentatively assumed to have the asymptotic form

$$v^\varepsilon(\boldsymbol{\xi}) = \varepsilon^{d_\Gamma} U(\bar{\boldsymbol{\xi}}) \quad (\boldsymbol{\xi} \in \Gamma_\varepsilon)\,, \qquad (v^\varepsilon, v^\varepsilon_{,n}) = \varepsilon^{d_\mathrm{s}}\big(V(\boldsymbol{\xi}), W(\boldsymbol{\xi})\big) \quad (\boldsymbol{\xi} \in S) \qquad (18)$$

where the leading orders d_Γ and d_s of the expansions are to be determined, and having introduced the scaled position vector $\bar{\boldsymbol{\xi}}$ defined by

$$\boldsymbol{\xi} = \boldsymbol{x}^\circ + \varepsilon\bar{\boldsymbol{\xi}} \qquad (19)$$

For further reference, note that the above definition implies

$$\mathrm{d}\Gamma_\xi = \varepsilon^2\,\mathrm{d}\Gamma_{\bar\xi} \quad (\boldsymbol{\xi} \in \Gamma_\varepsilon,\ \bar{\boldsymbol{\xi}} \in \mathscr{S}), \qquad \mathrm{d}V_\xi = \varepsilon^3\,\mathrm{d}V_{\bar\xi} \quad (\boldsymbol{\xi} \in B_\varepsilon,\ \bar{\boldsymbol{\xi}} \in \mathscr{B}) \qquad (20)$$

To determine the leading orders d_Γ, d_s, governing integral equations for $U(\bar{\boldsymbol{\xi}})$ and $V(\boldsymbol{\xi})$ are first sought. From (20) and on observing that

$$\begin{aligned} u(\boldsymbol{\xi}) &= u(\boldsymbol{x}^\circ) + O(\varepsilon) & G(\boldsymbol{x},\boldsymbol{\xi}) &= G(\boldsymbol{x},\boldsymbol{x}^\circ) + O(\varepsilon) \\ \nabla u(\boldsymbol{\xi}) &= \nabla u(\boldsymbol{x}^\circ) + O(\varepsilon) & \nabla G(\boldsymbol{x},\boldsymbol{\xi}) &= \nabla G(\boldsymbol{x},\boldsymbol{x}^\circ) + O(\varepsilon) \end{aligned} \qquad (\boldsymbol{x} \in S, \boldsymbol{\xi} \in \Gamma_\varepsilon)$$

the leading contributions in integral equation (15) for $\boldsymbol{x} \in S$ take the form

$$\varepsilon^{d_\mathrm{s}}\big[\mathcal{L}_{SS}(V,W)\big](\boldsymbol{x}) + \varepsilon^{d_\Gamma+2}\big[\mathcal{L}_{S\mathscr{S}}U\big](\boldsymbol{x}) = O(\varepsilon^3) \qquad (21)$$

where the integral operators $\mathcal{L}_{SS}$ and $\mathcal{L}_{S\mathscr{S}}$, defined by

$$\big[\mathcal{L}_{SS}(f,g)\big](\boldsymbol{x}) = \frac{1}{2}f(\boldsymbol{x}) + \int_{S_\mathrm{N}} H(\boldsymbol{x},\boldsymbol{\xi})f(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi - \int_{S_\mathrm{D}} G(\boldsymbol{x},\boldsymbol{\xi})g(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi \qquad (22)$$

$$\big[\mathcal{L}_{S\mathscr{S}}f\big](\boldsymbol{x}) = \nabla G(\boldsymbol{x},\boldsymbol{x}^\circ)\cdot\Big(\int_{\mathscr{S}} f(\bar{\boldsymbol{\xi}})\boldsymbol{n}(\bar{\boldsymbol{\xi}})\,\mathrm{d}\Gamma_{\bar\xi}\Big) \qquad (23)$$

do not depend on ε.

To determine the leading contribution of integral equation (15) for $\boldsymbol{x} \in \Gamma_\varepsilon$, it is useful to introduce further scaled geometric quantities:

$$\boldsymbol{x} = \varepsilon\bar{\boldsymbol{x}}\,, \quad \boldsymbol{r} = \varepsilon\bar{\boldsymbol{r}}\,, \quad r = \varepsilon\bar{r} \qquad (\boldsymbol{x},\boldsymbol{\xi} \in \Gamma_\varepsilon) \qquad (24)$$

One then has

$$H(\boldsymbol{x},\boldsymbol{\xi}) = \frac{1}{\varepsilon^2}H^0(\bar{\boldsymbol{x}},\bar{\boldsymbol{\xi}}) + O(1) \qquad (\boldsymbol{x},\boldsymbol{\xi} \in \Gamma_\varepsilon)$$

where

$$H^0(\bar{\boldsymbol{x}},\bar{\boldsymbol{\xi}}) = -\frac{(\bar{\boldsymbol{r}}.\boldsymbol{n})}{4\pi\bar{r}^3}$$

is the normal derivative of the fundamental solution for the static counterpart of the Laplace equation, and

$$G(\boldsymbol{x},\boldsymbol{\xi}) = G(\boldsymbol{x}^\circ,\boldsymbol{\xi}) + O(\varepsilon)\,, \quad \nabla G(\boldsymbol{x},\boldsymbol{\xi}) = \nabla G(\boldsymbol{x}^\circ,\boldsymbol{\xi}) + O(\varepsilon) \qquad (\boldsymbol{x} \in \Gamma_\varepsilon, \boldsymbol{\xi} \in S)$$

The leading contributions in integral equation (15) for $\boldsymbol{x} \in \Gamma_\varepsilon$ therefore take the form

$$\varepsilon^{d_\mathrm{s}}\big[\mathcal{L}_{\mathscr{S}S}(V,W)\big](\bar{\boldsymbol{x}}) + \varepsilon^{d_\Gamma}\big[\mathcal{L}_{\mathscr{S}\mathscr{S}}U\big](\bar{\boldsymbol{x}}) = O(\varepsilon) \qquad (\bar{\boldsymbol{x}} \in \mathscr{S}) \qquad (25)$$

where the integral operators $\mathcal{L}_{\mathscr{S}\mathscr{S}}$ and $\mathcal{L}_{\mathscr{S}S}$, defined by

$$\big[\mathcal{L}_{\mathscr{S}\mathscr{S}}f\big](\bar{\boldsymbol{x}}) = \frac{1}{2}f(\bar{\boldsymbol{x}}) + \int_{\mathscr{S}} H^0(\bar{\boldsymbol{x}},\bar{\boldsymbol{\xi}})f(\bar{\boldsymbol{\xi}})\,\mathrm{d}\Gamma_{\bar\xi} \qquad (\bar{\boldsymbol{x}} \in \mathscr{S}) \qquad (26)$$

$$\big[\mathcal{L}_{\mathscr{S}S}(f,g)\big](\bar{\boldsymbol{x}}) = \int_{S_\mathrm{N}} H(\boldsymbol{x}^\circ,\boldsymbol{\xi})f(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi - \int_{S_\mathrm{D}} G(\boldsymbol{x}^\circ,\boldsymbol{\xi})g(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi \qquad (27)$$

do not depend on ε. Note that $\mathcal{L}_{\mathscr{S}\mathscr{S}}$ is the integral operator associated with exterior Neumann problems over the region $\mathbb{R}^3\backslash\overline{\mathscr{B}}$ for the Laplace equation.

Now, d_s and d_Γ must be chosen so that nonzero solutions (U, V) are permitted in eqns (21), (25). Equation (25) then requires that $\min(d_\mathrm{s}, d_\Gamma) = 1$. Besides, since eqn.(21) with $d_\mathrm{s} = 1$ implies that $V = 0$, one must have $d_\Gamma = 1$.

Then, the lowest possible value for d_s is $d_s = 3$, and

$$v^\varepsilon(\boldsymbol{x}) = O(\varepsilon) \quad (\text{on } \Gamma_\varepsilon) , \qquad v^\varepsilon(\boldsymbol{x}) = O(\varepsilon^3) \quad (\text{on } S) \tag{28}$$

By virtue of the representation formula (17), the asymptotic behaviour of the scattered field on the observation surface is therefore

$$v^\varepsilon(\boldsymbol{x}) = O(\varepsilon^3) \quad (\text{on } S^{\text{obs}}) \tag{29}$$

As a consequence of (29), eqn.(14) provides a basis for expanding $J(\varepsilon, \boldsymbol{x}^\circ)$ up to the order $O(\varepsilon^6)$. This in turn requires expanding the traces on S^{obs} and Γ_ε of the scattered field to orders $O(\varepsilon^3)$ and $O(\varepsilon^4)$, respectively. It is worth noting that it is also possible to obtain the expansion of $J(\varepsilon, \boldsymbol{x}^\circ)$ up to the order $O(\varepsilon^6)$ on the basis of (12). However, this variant approach requires the expansion of v^ε on S^{obs} to order $O(\varepsilon^6)$, e.g. using the techniques expounded in [1], instead of $O(\varepsilon^3)$ when the adjoint solution is used.

Determination of the expansion of the scattered field As a result of the foregoing analysis, the traces on Γ_ε and S of the scattered field are now *a priori* sought in the form of N-term expansions

$$v^\varepsilon(\boldsymbol{\xi}) = \sum_{a=1}^{N} \varepsilon^a U_a(\bar{\boldsymbol{\xi}}) + o(\varepsilon^N) \qquad (\boldsymbol{\xi} \in \Gamma_\varepsilon) \tag{30}$$

$$\big(v^\varepsilon(\boldsymbol{\xi}), v^\varepsilon_{,n}(\boldsymbol{\xi}')\big) = \sum_{a=1}^{N} \varepsilon^{a+2} \big(V_a(\boldsymbol{\xi}), W_a(\boldsymbol{\xi}')\big) + o(\varepsilon^{N+2}) \qquad (\boldsymbol{\xi} \in S_{\text{N}}, \boldsymbol{\xi}' \in S_{\text{D}}) \tag{31}$$

Governing integral equations for the U_a and (V_a, W_a) are sought, by means of the following procedure: substitute expansions (30) and (31) into eqn.(15), expand the integrals about $\varepsilon = 0$, and equate to zero the sum of all contributions of like powers of $O(\varepsilon)$.

The Taylor expansions of the free-field and its gradient about $\boldsymbol{x}^\circ$, given by

$$\begin{aligned} u(\boldsymbol{\xi}) &= f^{(0)} + f^{(1)}(\bar{\boldsymbol{\xi}})\varepsilon + f^{(2)}(\bar{\boldsymbol{\xi}})\frac{\varepsilon^2}{2} + f^{(3)}(\bar{\boldsymbol{\xi}})\frac{\varepsilon^3}{6} + o(\varepsilon^3) \\ \nabla u(\boldsymbol{\xi}) &= \boldsymbol{g}^{(0)} + \boldsymbol{g}^{(1)}(\bar{\boldsymbol{\xi}})\varepsilon + \boldsymbol{g}^{(2)}(\bar{\boldsymbol{\xi}})\frac{\varepsilon^2}{2} + \boldsymbol{g}^{(3)}(\bar{\boldsymbol{\xi}})\frac{\varepsilon^3}{6} + o(\varepsilon^3) \end{aligned} \qquad (\boldsymbol{\xi} \in B_\varepsilon) \tag{32}$$

will play an essential role. In (32), $f^{(k)}(\bar{\boldsymbol{\xi}})$ and $\boldsymbol{g}^{(k)}(\bar{\boldsymbol{\xi}})$ are shorthand notations associated with the free-field and its derivatives at $\boldsymbol{x}^\circ$, defined by

$$\begin{aligned} f^{(0)} &= u(\boldsymbol{x}^\circ) & \boldsymbol{g}^{(0)} &= \nabla u(\boldsymbol{x}^\circ) \\ f^{(k)}(\bar{\boldsymbol{\xi}}) &= \nabla^{(k)} u(\boldsymbol{x}^\circ) \cdot (\underbrace{\bar{\boldsymbol{\xi}} \otimes \ldots \otimes \bar{\boldsymbol{\xi}}}_{k \text{ times}}) & \boldsymbol{g}^{(k)}(\bar{\boldsymbol{\xi}}) &= \nabla^{(k+1)} u(\boldsymbol{x}^\circ) \cdot (\underbrace{\bar{\boldsymbol{\xi}} \otimes \ldots \otimes \bar{\boldsymbol{\xi}}}_{k \text{ times}}) \end{aligned} \qquad (k \geq 1) \tag{33}$$

where $\nabla^{(k)} u(\boldsymbol{x}^\circ)$ denotes the k-th order tensor obtained by applying k times the gradient operator to the scalar function u and evaluating the result at $\boldsymbol{\xi} = \boldsymbol{x}^\circ$, and '$\cdot$' indicates in context the inner product over k indices.

Also essential, of course, are expansions of the fundamental kernels about $\varepsilon = 0$. Since their form depends on whether the collocation point $\boldsymbol{x}$ lies on Γ_ε or on S, a separate treatment of each case is warranted.

Case $\boldsymbol{x} \in \Gamma_\varepsilon$. The fundamental kernels about $\varepsilon = 0$ admit the expansions:

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{\xi}) &= \frac{1}{4\pi\bar{r}}\frac{1}{\varepsilon} + \frac{ik}{4\pi} - \frac{k^2\bar{r}}{8\pi}\varepsilon - \frac{ik^3\bar{r}^2}{24\pi}\varepsilon^2 + o(\varepsilon^2) \\ \nabla G(\boldsymbol{x}, \boldsymbol{\xi}) &= -\Big[\frac{1}{4\pi\bar{r}^3}\frac{1}{\varepsilon^2} + \frac{k^2}{8\pi\bar{r}} + \frac{ik^3}{12\pi}\varepsilon\Big]\bar{\boldsymbol{r}} + o(\varepsilon^2) \end{aligned} \qquad (\boldsymbol{x}, \boldsymbol{\xi} \in \Gamma_\varepsilon) \tag{34}$$

and (using the fact that $\nabla_x G(\boldsymbol{x}, \boldsymbol{\xi}) = -\nabla G(\boldsymbol{x}, \boldsymbol{\xi})$)

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{\xi}) &= G(\boldsymbol{x}^\circ, \boldsymbol{\xi}) - \big[\nabla G(\boldsymbol{x}^\circ, \boldsymbol{\xi}).\bar{\boldsymbol{x}}\big]\varepsilon + o(\varepsilon) \\ \nabla G(\boldsymbol{x}, \boldsymbol{\xi}) &= \nabla G(\boldsymbol{x}^\circ, \boldsymbol{\xi}) - \big[\nabla^{(2)} G(\boldsymbol{x}^\circ, \boldsymbol{\xi}).\bar{\boldsymbol{x}}\big]\varepsilon + o(\varepsilon) \end{aligned} \qquad (\boldsymbol{x} \in \Gamma_\varepsilon, \boldsymbol{\xi} \in S) \tag{35}$$

On substituting (20), (24), (30), (31), (32) and (34), (35) into (15), the contributions of orders $O(\varepsilon)$, $O(\varepsilon^2)$, $O(\varepsilon^3)$ and $O(\varepsilon^4)$ give rise to the following integral equations:

Order $O(\varepsilon)$: $$[\mathcal{L}_{\mathscr{S}\mathscr{S}} U_1](\bar{\boldsymbol{x}}) = -\int_{\mathscr{B}} \frac{1}{4\pi\bar{r}^3}\big[\boldsymbol{g}^{(0)}.\bar{\boldsymbol{r}}\big] \, dV_{\bar{\xi}} \tag{36}$$

Order $O(\varepsilon^2)$: $$[\mathcal{L}_{\mathscr{S}\mathscr{S}} U_2](\bar{\boldsymbol{x}}) = -\int_{\mathscr{B}} \frac{1}{4\pi\bar{r}^3}\big[\boldsymbol{g}^{(1)}(\bar{\boldsymbol{\xi}}).\bar{\boldsymbol{r}}\big] \, dV_{\bar{\xi}} - k^2 f^{(0)} \int_{\mathscr{B}} \frac{1}{4\pi\bar{r}} \, dV_{\bar{\xi}} \tag{37}$$

Order $O(\varepsilon^3)$: $$[\mathcal{L}_{\mathscr{S}\mathscr{S}}U_3](\bar{\boldsymbol{x}}) = -\int_{\mathscr{B}} \frac{1}{8\pi\bar{r}^3}\big[\boldsymbol{g}^{(2)}(\bar{\boldsymbol{\xi}}).\bar{\boldsymbol{r}}\,\big]\,\mathrm{d}V_{\bar{\xi}} - \int_{\mathscr{B}} \frac{k^2}{8\pi\bar{r}}\big[\boldsymbol{g}^{(0)}.\bar{\boldsymbol{r}} + 2f^{(1)}(\bar{\boldsymbol{\xi}})\,\big]\,\mathrm{d}V_{\bar{\xi}} - f^{(0)}\int_{\mathscr{B}}\frac{\mathrm{i}k^3}{4\pi}\,\mathrm{d}V_{\bar{\xi}} + \int_{\mathscr{S}} U_1(\bar{\boldsymbol{\xi}})\frac{k^2}{8\pi\bar{r}}(\bar{\boldsymbol{r}}.\boldsymbol{n}(\bar{\boldsymbol{\xi}}))\,\mathrm{d}\Gamma_{\bar{\xi}} + D_1 \tag{38}$$

Order $O(\varepsilon^4)$: $$\begin{aligned}[\mathcal{L}_{\mathscr{S}\mathscr{S}}U_4](\bar{\boldsymbol{x}}) = &-\int_{\mathscr{B}} \frac{1}{24\pi\bar{r}^3}\big[\boldsymbol{g}^{(3)}(\bar{\boldsymbol{\xi}}).\bar{\boldsymbol{r}}\,\big]\,\mathrm{d}V_{\bar{\xi}} - \int_{\mathscr{B}} \frac{k^2}{8\pi\bar{r}}\big[\boldsymbol{g}^{(1)}(\bar{\boldsymbol{\xi}}).\bar{\boldsymbol{r}} + f^{(2)}(\bar{\boldsymbol{\xi}})\,\big]\,\mathrm{d}V_{\bar{\xi}} \\ &- \int_{\mathscr{B}}\frac{\mathrm{i}k^3}{12\pi}\big[\boldsymbol{g}^{(0)}.\bar{\boldsymbol{r}} + 3f^{(1)}(\bar{\boldsymbol{\xi}})\,\big]\,\mathrm{d}V_{\bar{\xi}} + f^{(0)}\int_{\mathscr{B}}\frac{k^2}{8\pi}\bar{r}\,\mathrm{d}V_{\bar{\xi}} \\ &+ \int_{\mathscr{S}}\Big[U_2(\bar{\boldsymbol{\xi}})\frac{k^2}{8\pi\bar{r}} + U_1(\bar{\boldsymbol{\xi}})\frac{\mathrm{i}k^3}{12\pi}\Big](\bar{\boldsymbol{r}}.\boldsymbol{n}(\bar{\boldsymbol{\xi}}))\,\mathrm{d}\Gamma_{\bar{\xi}} + \boldsymbol{E}.\bar{\boldsymbol{x}} + D_2\end{aligned} \tag{39}$$

where the scalar constants $D_1(\boldsymbol{x}^{\circ})$, $D_2(\boldsymbol{x}^{\circ})$ and the vector constant $\boldsymbol{E}(\boldsymbol{x}^{\circ})$ are defined by

$$\begin{aligned}D_1(\boldsymbol{x}^{\circ}) &= -\int_{S_{\mathrm{N}}} H(\boldsymbol{x}^{\circ},\boldsymbol{\xi})V_1(\boldsymbol{\xi})\,\mathrm{d}\Gamma_{\xi} + \int_{S_{\mathrm{D}}} G(\boldsymbol{x}^{\circ},\boldsymbol{\xi})W_1(\boldsymbol{\xi})\,\mathrm{d}\Gamma_{\xi} \\ D_2(\boldsymbol{x}^{\circ}) &= -\int_{S_{\mathrm{N}}} H(\boldsymbol{x}^{\circ},\boldsymbol{\xi})V_2(\boldsymbol{\xi})\,\mathrm{d}\Gamma_{\xi} + \int_{S_{\mathrm{D}}} G(\boldsymbol{x}^{\circ},\boldsymbol{\xi})W_2(\boldsymbol{\xi})\,\mathrm{d}\Gamma_{\xi} \\ \boldsymbol{E}(\boldsymbol{x}^{\circ}) &= \int_{S_{\mathrm{N}}} \big[\,\nabla^{(2)}G(\boldsymbol{x}^{\circ},\boldsymbol{\xi}).\boldsymbol{n}(\boldsymbol{\xi})\,\big]V_1(\boldsymbol{\xi})\,\mathrm{d}\Gamma_{\xi} - \int_{S_{\mathrm{D}}} \nabla G(\boldsymbol{x}^{\circ},\boldsymbol{\xi})W_1(\boldsymbol{\xi})\,\mathrm{d}\Gamma_{\xi}\end{aligned} \tag{40}$$

Equations (36) and (37) can be solved for U_1 and U_2, respectively. Owing to the structure of the right-hand sides of (36) and (37), the solutions U_1 and U_2 can be cast in the form

$$U_1(\boldsymbol{\xi}) = \nabla u(\boldsymbol{x}^{\circ}).\boldsymbol{\mathcal{U}}_1(\boldsymbol{\xi}) \qquad U_2(\boldsymbol{\xi}) = \nabla^{(2)}u(\boldsymbol{x}^{\circ}):\boldsymbol{\mathcal{U}}_2^{(2)}(\boldsymbol{\xi}) + k^2u(\boldsymbol{x}^{\circ})\mathcal{U}_2^{(0)}(\boldsymbol{\xi}) \tag{41}$$

where the scalar function $\mathcal{U}_2^{(0)}$ and each component of the vector function $\boldsymbol{\mathcal{U}}_1$ and the symmetric tensor function $\boldsymbol{\mathcal{U}}_2^{(2)}$ solve exterior problems for the Laplace equation on the domain $\mathbb{R}^3\backslash\mathscr{B}$ and do not depend on $\boldsymbol{x}^{\circ}$.

In contrast, eqns (38) and (39) cannot be solved on a stand-alone basis, as they also involve the unknowns (V_1, W_1) and (V_2, W_2). Two additional equations involving (V_1, W_1) and (V_2, W_2) are therefore required. To set up such equations, the case $\boldsymbol{x} \in S$ (i.e. the collocation point located on the external surface) is now considered.

Case $\boldsymbol{x} \in S$. The fundamental kernels about $\varepsilon = 0$ admit the expansions:

$$\begin{aligned}G(\boldsymbol{x},\boldsymbol{\xi}) &= G(\boldsymbol{x},\boldsymbol{x}^{\circ}) + \varepsilon\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ})\cdot\bar{\boldsymbol{\xi}} + o(\varepsilon) \\ \nabla G(\boldsymbol{x},\boldsymbol{\xi}) &= \nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ}) + \varepsilon\nabla^{(2)}G(\boldsymbol{x},\boldsymbol{x}^{\circ})\cdot\bar{\boldsymbol{\xi}} + o(\varepsilon)\end{aligned} \qquad (\boldsymbol{x}\in S, \boldsymbol{\xi}\in\Gamma_{\varepsilon}) \tag{42}$$

On substituting (20), (30), (31) and (42) into (15), no contribution of order $O(\varepsilon)$ and $O(\varepsilon^2)$ arise. Collecting contributions of orders $O(\varepsilon^3)$ and $O(\varepsilon^4)$, one arrives at the following integral equations:

Order $O(\varepsilon^3)$: $$[\mathcal{L}_{SS}(V_1,W_1)](\boldsymbol{x}) = -\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ})\cdot\int_{\mathscr{S}} U_1(\bar{\boldsymbol{\xi}})\boldsymbol{n}(\bar{\boldsymbol{\xi}})\,\mathrm{d}\Gamma_{\bar{\xi}} + |\mathscr{B}|\,\big[\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ}).\nabla u(\boldsymbol{x}^{\circ}) - k^2G(\boldsymbol{x},\boldsymbol{x}^{\circ})u(\boldsymbol{x}^{\circ})\big] \tag{43}$$

Order $O(\varepsilon^4)$: $$\begin{aligned}&[\mathcal{L}_{SS}(V_2,W_2)](\boldsymbol{x}) + \Big(\int_{\mathscr{S}} U_1(\bar{\boldsymbol{\xi}})[\bar{\boldsymbol{\xi}}\otimes\boldsymbol{n}(\bar{\boldsymbol{\xi}})]\,\mathrm{d}\Gamma_{\bar{\xi}}\Big):\nabla^{(2)}G(\boldsymbol{x},\boldsymbol{x}^{\circ}) + \Big(\int_{\mathscr{S}} U_2(\bar{\boldsymbol{\xi}})\boldsymbol{n}(\bar{\boldsymbol{\xi}})\,\mathrm{d}\Gamma_{\bar{\xi}}\Big)\cdot\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ}) \\ &= \int_{\mathscr{B}}\Big\{\nabla\big[\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ}).\nabla u(\boldsymbol{x}^{\circ}) - k^2G(\boldsymbol{x},\boldsymbol{x}^{\circ})u(\boldsymbol{x}^{\circ})\big]\cdot\bar{\boldsymbol{\xi}}\,\mathrm{d}V_{\bar{\xi}} = 0\end{aligned} \tag{44}$$

where the operator $\mathcal{L}_{SS}$ is defined by (22), and $\boldsymbol{x}^{\circ}$ must (without loss of generality) be assumed to lie at the center of B_{ε} for the last equality in (44) to hold.

Once eqns (36) and (37) are solved for U_1 and U_2, the functions (V_1, W_1) and (V_2, W_2) can be obtained by solving eqns (43) and (44), respectively. On substituting U_1 and U_2 as expressed by (41), eqns (43) and (44) take the equivalent, more compact, form

$$\big[\mathcal{L}_{SS}(V_1,W_1)\big](\boldsymbol{x}) = |\mathscr{B}|\,\big[\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ}).\big[\,\mathbf{I}-\boldsymbol{\mathcal{A}}_1\,\big].\nabla u(\boldsymbol{x}^{\circ}) - k^2G(\boldsymbol{x},\boldsymbol{x}^{\circ})u(\boldsymbol{x}^{\circ})\big] \tag{45}$$

$$\begin{aligned}\big[\mathcal{L}_{SS}(V_2,W_2)\big](\boldsymbol{x}) = -\,|\mathscr{B}|\,\big[&\nabla u(\boldsymbol{x}^{\circ}).\boldsymbol{\mathcal{A}}_1':\nabla^{(2)}G(\boldsymbol{x},\boldsymbol{x}^{\circ}) + \nabla^{(2)}u(\boldsymbol{x}^{\circ}):\boldsymbol{\mathcal{A}}_2^{(2)}.\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ}) \\ &+ k^2u(\boldsymbol{x}^{\circ})\boldsymbol{\mathcal{A}}_2^{(0)}.\nabla G(\boldsymbol{x},\boldsymbol{x}^{\circ})\big]\end{aligned} \tag{46}$$

where $\mathcal{A}_1$, $\mathcal{A}_1'$, $\mathcal{A}_2^{(0)}$ and $\mathcal{A}_2^{(2)}$ are defined by

$$\mathcal{A}_1 = \frac{1}{|\mathscr{B}|}\int_{\mathscr{S}} [\mathcal{U}_1(\bar{\xi}) \otimes \boldsymbol{n}(\bar{\xi})] \, \mathrm{d}\Gamma_{\bar{\xi}} \qquad \mathcal{A}_1' = \frac{1}{|\mathscr{B}|}\int_{\mathscr{S}} [\mathcal{U}_1(\bar{\xi}) \otimes \bar{\xi} \otimes \boldsymbol{n}(\bar{\xi})] \, \mathrm{d}\Gamma_{\bar{\xi}}$$
$$\mathcal{A}_2^{(0)} = \frac{1}{|\mathscr{B}|}\int_{\mathscr{S}} [\mathcal{U}_2^{(0)}(\bar{\xi})\, \boldsymbol{n}(\bar{\xi})] \, \mathrm{d}\Gamma_{\bar{\xi}} \qquad \mathcal{A}_2^{(2)} = \frac{1}{|\mathscr{B}|}\int_{\mathscr{S}} [\mathcal{U}_2^{(2)}(\bar{\xi}) \otimes \boldsymbol{n}(\bar{\xi})] \, \mathrm{d}\Gamma_{\bar{\xi}} \tag{47}$$

The functions (V_1, W_1) and (V_2, W_2) therefore appear to be the traces on S of solutions to the Helmholtz equation generated by point sources applied at $\boldsymbol{x}^\circ$. In particular, since the right-hand sides of eqns (45), (46) depend on $\boldsymbol{x}^\circ$, the functions (V_1, W_1) and (V_2, W_2) also depend on $\boldsymbol{x}^\circ$ as a parameter. Notations such as $V_1(\boldsymbol{\xi}, \boldsymbol{x}^\circ)$ will be used to emphasize that fact.

2.3. Expansion of the cost function: obstacle of arbitrary shape

The expansion in powers of ε of the cost function $J(\varepsilon, \boldsymbol{x}^\circ)$ is now set up on the basis of (14) and the results obtained in the previous section. Substituting (31) into the integral over S^{obs}, and (20) and expansions (30) for v^ε and (32) for u and $\hat{u}$ into the integrals over Γ_ε leads to an explicit form of expansion (14), up to order $O(\varepsilon^6)$:

$$J(\varepsilon, \boldsymbol{x}^\circ) = J(0) + A_3(\boldsymbol{x}^\circ)\varepsilon^3 + A_4(\boldsymbol{x}^\circ)\varepsilon^4 + A_5(\boldsymbol{x}^\circ)\varepsilon^5 + A_6(\boldsymbol{x}^\circ)\varepsilon^6 + o(\varepsilon^6) = J(0) + J^{(6)}(\varepsilon; \boldsymbol{x}^\circ) + o(\varepsilon^6) \tag{48}$$

which defines the polynomial (in ε) $J^{(6)}(\varepsilon; \boldsymbol{x}^\circ)$ (note that this definition implies $J^{(6)}(0; \boldsymbol{x}^\circ) = 0$), and where the coefficients $A_3(\boldsymbol{x}^\circ)$, $A_4(\boldsymbol{x}^\circ)$, $A_5(\boldsymbol{x}^\circ)$ and $A_6(\boldsymbol{x}^\circ)$ are found to be given by

$$A_3(\boldsymbol{x}^\circ) = |\mathscr{B}| \left[\nabla u.(\boldsymbol{I} - \mathcal{A}_1).\nabla\hat{u} - k^2 u\hat{u}\right](\boldsymbol{x}^\circ) \tag{49}$$

$$A_4(\boldsymbol{x}^\circ) = -|\mathscr{B}| \left[\nabla u.\mathcal{A}_1' : \nabla^{(2)}\hat{u} + \nabla^{(2)}u : \mathcal{A}_2^{(2)}.\nabla\hat{u} + k^2 u(\mathcal{A}_2^{(0)}.\nabla\hat{u})\right](\boldsymbol{x}^\circ) \tag{50}$$

$$\begin{aligned} A_5(\boldsymbol{x}^\circ) = &-\frac{1}{2}\int_{\mathscr{S}} \left[U_1(\bar{\xi})\hat{\boldsymbol{g}}^{(2)}(\bar{\xi}) + 2U_2(\bar{\xi})\hat{\boldsymbol{g}}^{(1)}(\bar{\xi}) + 2U_3(\bar{\xi})\hat{\boldsymbol{g}}^{(0)}\right] \cdot \boldsymbol{n}(\bar{\xi}) \, \mathrm{d}\Gamma_{\xi} \\ &+ \frac{1}{2}\left[\nabla^{(3)}u.\nabla\hat{u} + 2\nabla^{(2)}u.\nabla^{(2)}\hat{u} + \nabla^{(3)}\hat{u}.\nabla u\right](\boldsymbol{x}^\circ) : \mathcal{I}_2(\mathscr{B}) \\ &- \frac{k^2}{2}\left[u\nabla^{(2)}\hat{u} + 2\nabla u \otimes \nabla\hat{u} + \hat{u}\nabla^{(2)}u\right](\boldsymbol{x}^\circ) : \mathcal{I}_2(\mathscr{B}) \end{aligned} \tag{51}$$

$$\begin{aligned} A_6(\boldsymbol{x}^\circ) = &\frac{1}{2}\int_{S^{\text{obs}}} \frac{\partial^2\varphi}{\partial u^2}(u(\boldsymbol{\xi}))\, V_1^2(\boldsymbol{\xi}; \boldsymbol{x}^\circ) \, \mathrm{d}\Gamma_\xi \\ &- \frac{1}{6}\int_{\mathscr{S}} \left[U_1(\bar{\xi})\hat{\boldsymbol{g}}^{(3)}(\bar{\xi}) + 3U_2(\bar{\xi})\hat{\boldsymbol{g}}^{(2)}(\bar{\xi}) + 6U_3(\bar{\xi})\hat{\boldsymbol{g}}^{(1)}(\bar{\xi}) + 6U_4(\bar{\xi})\hat{\boldsymbol{g}}^{(0)}\right] \cdot \boldsymbol{n}(\bar{\xi}) \, \mathrm{d}\Gamma_\xi \\ &+ \frac{1}{6}\left[\nabla^{(4)}u.\nabla\hat{u} + 3\nabla^{(3)}u.\nabla^{(2)}\hat{u} + 3\nabla^{(2)}u.\nabla^{(3)}\hat{u} + \nabla u.\nabla^{(4)}\hat{u}\right](\boldsymbol{x}^\circ) \cdot \mathcal{I}_3(\mathscr{B}) \\ &- \frac{k^2}{6}\left[u\nabla^{(3)}\hat{u} + 3\nabla u \otimes \nabla^{(2)}\hat{u} + 3\nabla^{(2)}u \otimes \nabla\hat{u} + k^2\hat{u}\nabla^{(3)}u\right](\boldsymbol{x}^\circ) \cdot \mathcal{I}_3(\mathscr{B}) \end{aligned} \tag{52}$$

with the shorthand notations $\hat{g}^{(k)}$ and $\hat{\boldsymbol{g}}^{(k)}$ defined by (33) but in terms of the adjoint field $\hat{u}$ instead of the free field u, and with

$$\mathcal{I}_2(\mathscr{B}) = \int_{\mathscr{B}} (\bar{\xi} \otimes \bar{\xi}) \, \mathrm{d}V_{\bar{\xi}} \qquad \mathcal{I}_3(\mathscr{B}) = \int_{\mathscr{B}} (\bar{\xi} \otimes \bar{\xi} \otimes \bar{\xi}) \, \mathrm{d}V_{\bar{\xi}} \tag{53}$$

(note that $\mathcal{I}_2(\mathscr{B})$ is the geometrical inertia tensor of the normalized obstacle $\mathscr{B}$). Note that the third-order tensor $\mathcal{I}_3(\mathscr{B})$ vanishes for all shapes such that $\mathscr{B}$ has central symmetry (i.e. if $\bar{\boldsymbol{\xi}} \in \mathscr{B} \Leftrightarrow -\bar{\boldsymbol{\xi}} \in \mathscr{B}$), a property which is met by several simple shapes (spheres, ellipsoids and rectangular boxes, among others).

Coefficients $A_3(\boldsymbol{x}^\circ)$ and $A_4(\boldsymbol{x}^\circ)$ are local in that they involve the free and adjoint fields and their gradients at the grid point $\boldsymbol{x}^\circ$ only. On the other hand, coefficients $A_5(\boldsymbol{x}^\circ)$ and $A_6(\boldsymbol{x}^\circ)$ are non-local because the solutions U_3 and U_4 are themselves non-local. Moreover, $A_3(\boldsymbol{x}^\circ) = |\mathscr{B}|\, \mathcal{T}(\boldsymbol{x}^\circ)$, where $\mathcal{T}(\boldsymbol{x}^\circ)$ is the topological derivative defined by (5), and $\boldsymbol{I} - \mathcal{A}_1 = \boldsymbol{A}(\mathscr{S})$ where $\boldsymbol{A}(\mathscr{S})$ is the second-order tensor featured in (5).

2.4. Expansion of the cost function: spherical obstacle

The special case of a *spherical* rigid obstacle B_ε (for which $\mathscr{B}$ is the unit ball, $\mathscr{S}$ the unit sphere and $|\mathscr{B}| = 4\pi/3$), for which the analytical treatment can be carried out further, is now considered in detail.

First, the following relationships are established by means of analytic integrations:

$$[\mathcal{L}_{\mathscr{S}\mathscr{S}} f^{(0)}](\bar{x}) = f^{(0)}\,, \qquad [\mathcal{L}_{\mathscr{S}\mathscr{S}} f^{(1)}](\bar{x}) = \frac{2}{3} f^{(1)}(\bar{x}) \tag{54}$$

$$[\mathcal{L}_{\mathscr{S}\mathscr{S}} f^{(2)}](\bar{x}) = \frac{3}{5} f^{(2)}(\bar{x}) - \frac{2k^2}{15} f^{(0)}, \qquad [\mathcal{L}_{\mathscr{S}\mathscr{S}} f^{(3)}](\bar{x}) = \frac{4}{7} f^{(3)}(\bar{x}) - \frac{2k^2}{35} f^{(1)}(\bar{x}),$$
$$[\mathcal{L}_{\mathscr{S}\mathscr{S}} f^{(4)}](\bar{x}) = \frac{5}{9} f^{(4)}(\bar{x}) - \frac{4k^2}{105} f^{(2)}(\bar{x}) + \frac{8k^4}{105} f^{(0)} \tag{55}$$

where $f^{(0)}$ and $f^{(1)}(\bar{x})$ to $f^{(4)}(\bar{x})$ are again defined by (33). In identities (54), $f^{(0)}$ and $f^{(1)}(\bar{x})$ could be replaced by any constant and linear function of $\bar{x}$, respectively, whereas identities (55) require the relations

$$\boldsymbol{I}\!:\!\nabla^{(2+\ell)} u = -k^2 \nabla^{(\ell)} u \quad (\ell \geq 0), \qquad (\boldsymbol{I} \otimes \boldsymbol{I}) \cdot \nabla^{(4)} u = k^4 u, \tag{56}$$

which stem from the free-field u being a solution of the Helmholtz equation, to hold true.

Moreover, the right-hand sides of eqns (36) and (37) feature integrals involving the free-field and its derivatives at $\boldsymbol{x}^{\text{o}}$. Upon evaluating analytically these integrals, eqns (36) and (37) take the following explicit form:

$$[\mathcal{L}_{\mathscr{S}\mathscr{S}} U_1](\bar{x}) = \frac{1}{3} f^{(1)}(\bar{x}) \qquad [\mathcal{L}_{\mathscr{S}\mathscr{S}} U_2](\bar{x}) = -\frac{4}{15} f^{(0)} + \frac{1}{5} f^{(2)}(\bar{x})$$

In view of eqns (54) and (55), the solutions to these equations are then readily found to be given by

$$U_1(\bar{\xi}) = \frac{1}{2} f^{(1)}(\bar{\xi}) \qquad U_2(\bar{\xi}) = \frac{1}{3} f^{(2)}(\bar{\xi}) - \frac{2}{9} k^2 f^{(0)} \tag{57}$$

or equivalently, with reference to definition (41):

$$\boldsymbol{\mathcal{U}}_1(\bar{\xi}) = \frac{1}{2}\bar{\boldsymbol{\xi}} \qquad \mathcal{U}_2^0(\bar{\xi}) = -\frac{2}{9} \qquad \boldsymbol{\mathcal{U}}_2^2(\bar{\xi}) = \frac{1}{3}(\bar{\boldsymbol{\xi}} \otimes \bar{\boldsymbol{\xi}}) \tag{58}$$

Once U_1, U_2 are known, explicit expressions of the constant tensors $\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_1', \boldsymbol{\mathcal{A}}_2^0, \boldsymbol{\mathcal{A}}_2^2$, defined by (47), are obtained from (58):

$$\boldsymbol{\mathcal{A}}_1 = \frac{3}{8\pi} \int_{\mathscr{S}} [\bar{\boldsymbol{\xi}} \otimes \boldsymbol{n}(\bar{\boldsymbol{\xi}})] \, \mathrm{d}\Gamma_{\bar{\xi}} = -\frac{1}{2}\boldsymbol{I} \qquad \boldsymbol{\mathcal{A}}_1' = \frac{3}{8\pi} \int_{\mathscr{S}} [\bar{\boldsymbol{\xi}} \otimes \boldsymbol{n}(\bar{\boldsymbol{\xi}}) \otimes \bar{\boldsymbol{\xi}}] \, \mathrm{d}\Gamma_{\bar{\xi}} = \boldsymbol{0}$$
$$\boldsymbol{\mathcal{A}}_2^{(0)} = -\frac{1}{6\pi} \int_{\mathscr{S}} \boldsymbol{n}(\bar{\boldsymbol{\xi}}) \, \mathrm{d}\Gamma_{\bar{\xi}} = \boldsymbol{0} \qquad \boldsymbol{\mathcal{A}}_2^{(2)} = \frac{1}{4\pi} \int_{\mathscr{S}} [\bar{\boldsymbol{\xi}} \otimes \bar{\boldsymbol{\xi}} \otimes \boldsymbol{n}(\bar{\boldsymbol{\xi}})] \, \mathrm{d}\Gamma_{\bar{\xi}} = \boldsymbol{0} \tag{59}$$

having used symmetry considerations and the fact that the inward unit normal on $\mathscr{S}$ is $\boldsymbol{n}(\bar{\boldsymbol{\xi}}) = -\bar{\boldsymbol{\xi}}$.

On substituting (59) into (49) and (50), the coefficients $A_3(\boldsymbol{x}^{\text{o}})$ and $A_4(\boldsymbol{x}^{\text{o}})$ of expansion (48) are found to be given by

$$A_3(\boldsymbol{x}^{\text{o}}) = \big(2\pi \nabla u . \nabla \hat{u} - \frac{4\pi}{3} u\hat{u}\big)(\boldsymbol{x}^{\text{o}}), \qquad A_4(\boldsymbol{x}^{\text{o}}) = 0 \tag{60}$$

Besides, explicit expressions can now be derived for the right-hand sides of eqns (38) and (39). As a result, the governing equations for the remaining unknowns U_3, U_4 become:

$$[\mathcal{L}_{\mathscr{S}} U_3](\bar{x}) = \frac{1}{14} f^{(3)}(\bar{x}) + \frac{23}{210} k^2 f^{(1)}(\bar{x}) - \frac{ik^3}{3} f^{(0)} + D_1(\boldsymbol{x}^{\text{o}}) \tag{61}$$

$$[\mathcal{L}_{\mathscr{S}} U_4](\bar{x}) = \boldsymbol{E}.\bar{\boldsymbol{x}} + D_2(\boldsymbol{x}^{\text{o}}) + \frac{ik^3}{6} f^{(1)}(\bar{x}) + \frac{1}{54} f^{(4)}(\bar{x}) + \frac{k^2}{105} f^{(2)}(\bar{x}) + \frac{346k^4}{945} f^{(0)} \tag{62}$$

where the scalar constants $D_1(\boldsymbol{x}^{\text{o}})$, $D_2(\boldsymbol{x}^{\text{o}})$ and the vector constant $\boldsymbol{E}(\boldsymbol{x}^{\text{o}})$ are again defined by (40), and the other integrals featured in eqns (38) and (39) could be evaluated analytically. Again, in view of identities (54) and (55) and the form assumed by the right-hand sides of (61) and (62), the solutions U_3, U_4 are readily found to be

$$U_3(\bar{\xi}) = \frac{1}{8} f^{(3)}(\bar{\xi}) + \frac{7k^2}{40} f^{(1)}(\bar{\xi}) - \frac{ik^3}{3} f^{(0)} + D_1(\boldsymbol{x}^{\text{o}})$$
$$U_4(\bar{\xi}) = \frac{1}{30} f^{(4)}(\bar{\xi}) + \frac{17}{945} f^{(2)}(\bar{\xi}) + \frac{5188}{14175} f^{(0)} + D_2(\boldsymbol{x}^{\text{o}}) + \frac{3}{2}\boldsymbol{E}(\boldsymbol{x}^{\text{o}}).\bar{\boldsymbol{\xi}} + \frac{ik^3}{4} f^{(1)}(\bar{\xi}) \tag{63}$$

To establish explicit formulae for coefficients $A_5(\boldsymbol{x}^{\text{o}})$ and $A_6(\boldsymbol{x}^{\text{o}})$, expressions (57) of U_1, U_2 and (63) of U_3, U_4 are now substituted into (51), (52). One obtains at first

$$\begin{aligned} A_5(\boldsymbol{x}^{\text{o}}) = {} & \Big[\Big(\frac{ik^3}{3} u - D_1 \Big) \nabla \hat{u} \Big](\boldsymbol{x}^{\text{o}}) \cdot \boldsymbol{\mathcal{I}}_0(\mathscr{S}) + \Big[\frac{2k^2}{9} u \nabla^{(2)} \hat{u} - \frac{7k^2}{40} (\nabla u \otimes \nabla \hat{u}) \Big](\boldsymbol{x}^{\text{o}}) \cdot \boldsymbol{\mathcal{I}}_1(\mathscr{S}) \\ & - \Big[\frac{1}{4} (\nabla u \otimes \nabla^{(3)} \hat{u} + \frac{1}{3} (\nabla^{(2)} u \otimes \nabla^{(2)} \hat{u}) + \frac{1}{8} (\nabla^{(3)} u \otimes \nabla \hat{u}) \Big](\boldsymbol{x}^{\text{o}}) \cdot \boldsymbol{\mathcal{I}}_3(\mathscr{S}) \\ & + \frac{1}{2} \Big[\nabla^{(3)} u . \nabla \hat{u} + 2 \nabla^{(2)} u . \nabla^{(2)} \hat{u} + \nabla^{(3)} \hat{u} . \nabla u \Big](\boldsymbol{x}^{\text{o}}) \!:\! \boldsymbol{\mathcal{I}}_2(\mathscr{B}) \\ & - \frac{k^2}{2} \Big[u \nabla^{(2)} \hat{u} + 2 \nabla u \otimes \nabla \hat{u} + \hat{u} \nabla^{(2)} u \Big](\boldsymbol{x}^{\text{o}}) \!:\! \boldsymbol{\mathcal{I}}_2(\mathscr{B}) \end{aligned} \tag{64}$$

$$A_6(\boldsymbol{x}^\circ) = \frac{1}{2}\int_{S^{\text{obs}}} \frac{\partial^2 \varphi}{\partial u^2}(u(\boldsymbol{\xi}))\, V_1^2(\boldsymbol{\xi};\boldsymbol{x}^\circ)\, \mathrm{d}\Gamma_\xi - \Big[\Big(D_2 + \frac{5188}{14175}u\Big)\nabla\hat{u}\Big](\boldsymbol{x}^\circ)\cdot\boldsymbol{\mathcal{I}}_0(\mathscr{S})$$
$$+\Big[\Big(\frac{ik^3}{3}u - D_1\Big)\nabla^{(2)}\hat{u} - \Big(\frac{ik^3}{4}\nabla u + \frac{3}{2}\boldsymbol{E}\Big)\otimes\nabla\hat{u}\Big](\boldsymbol{x}^\circ)\cdot\boldsymbol{\mathcal{I}}_1(\mathscr{S})$$
$$+\Big[\frac{1}{6}u\nabla^{(3)}\hat{u} - \frac{7k^2}{40}(\nabla u\otimes\nabla^{(2)}\hat{u}) - \frac{17}{945}(\nabla^{(2)}u\otimes\nabla\hat{u})\Big](\boldsymbol{x}^\circ)\cdot\boldsymbol{\mathcal{I}}_2(\mathscr{S})$$
$$-\Big[\frac{1}{12}(\nabla u\otimes\nabla^{(4)}\hat{u}) + \frac{1}{6}(\nabla^{(2)}u\otimes\nabla^{(3)}\hat{u}) + \frac{1}{8}(\nabla^{(3)}u\otimes\nabla^{(2)}\hat{u}) + \frac{1}{30}(\nabla^{(4)}u\otimes\nabla\hat{u})\Big](\boldsymbol{x}^\circ)\cdot\boldsymbol{\mathcal{I}}_4(\mathscr{S})$$
$$+\frac{1}{6}\Big[\nabla^{(4)}u.\nabla\hat{u} + 3\nabla^{(3)}u.\nabla^{(2)}\hat{u} + 3\nabla^{(2)}u.\nabla^{(3)}\hat{u} + \nabla u.\nabla^{(4)}\hat{u}\Big](\boldsymbol{x}^\circ)\cdot\boldsymbol{\mathcal{I}}_3(\mathscr{B})$$
$$-\frac{k^2}{6}\Big[u\nabla^{(3)}\hat{u} + 3\nabla u\otimes\nabla^{(2)}\hat{u} + 3\nabla^{(2)}u\otimes\nabla\hat{u} + k^2\hat{u}\nabla^{(3)}u\Big](\boldsymbol{x}^\circ)\cdot\boldsymbol{\mathcal{I}}_3(\mathscr{B}) \qquad (65)$$

having defined the $\ell+1$-th order tensor $\boldsymbol{\mathcal{I}}_\ell(\mathscr{S})$ by

$$\boldsymbol{\mathcal{I}}_\ell(\mathscr{S}) = \int_{\mathscr{S}} \underbrace{(\bar{\boldsymbol{\xi}}\otimes\ldots\otimes\bar{\boldsymbol{\xi}})}_{\ell\text{ times}}\otimes\boldsymbol{n}(\bar{\boldsymbol{\xi}})\,\mathrm{d}\Gamma_{\bar{\xi}} \qquad (0\le\ell\le 4) \qquad (66)$$

Since $\mathscr{B}$ is the unit ball and $\mathscr{S}$ the unit sphere, one readily finds that

$$\boldsymbol{\mathcal{I}}_2(\mathscr{B}) = \frac{4\pi}{15}\boldsymbol{I}, \qquad \boldsymbol{\mathcal{I}}_3(\mathscr{B}) = \boldsymbol{0}$$
$$\boldsymbol{\mathcal{I}}_0(\mathscr{S}) = \boldsymbol{0}, \qquad \boldsymbol{\mathcal{I}}_1(\mathscr{S}) = -\frac{4\pi}{3}\boldsymbol{I}, \qquad \boldsymbol{\mathcal{I}}_2(\mathscr{S}) = \boldsymbol{0}, \qquad \boldsymbol{\mathcal{I}}_3(\mathscr{S}) = -\frac{4\pi}{15}\boldsymbol{J}, \qquad \boldsymbol{\mathcal{I}}_4(\mathscr{S}) = \boldsymbol{0} \qquad (67)$$

where the fourth-order tensor $\boldsymbol{J}$ is defined by $J_{ijk\ell} = \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}$. Also, since both u and $\hat{u}$ solve the Helmholtz equation, they are readily shown to verify

$$\boldsymbol{I}:(\nabla^{(3)}u.\nabla\hat{u}) = -k^2(\nabla u.\nabla\hat{u}),$$
$$\boldsymbol{J}\cdot(\nabla^{(3)}u\otimes\nabla\hat{u}) = -3k^2(\nabla u.\nabla\hat{u}), \qquad \boldsymbol{J}\cdot(\nabla^{(2)}u\otimes\nabla^{(2)}\hat{u}) = k^4u\hat{u} + 2(\nabla^{(2)}u:\nabla^{(2)}\hat{u}) \qquad (68)$$

in addition to the already known set of identities (56).

With the help of identities (56), (67) and (68), formulae (64) and (65) yield the following explicit expressions for $A_5(\boldsymbol{x}^\circ)$ and $A_6(\boldsymbol{x}^\circ)$:

$$A_5(\boldsymbol{x}^\circ) = \Big[\frac{4\pi}{9}\nabla\nabla u:\nabla\nabla\hat{u} - \frac{3\pi}{5}k^2\nabla u.\nabla\hat{u} + \frac{88\pi}{135}k^4u\hat{u}\Big](\boldsymbol{x}^\circ) \qquad (69)$$
$$A_6(\boldsymbol{x}^\circ) = \Big[\Big(\frac{4\pi}{9}ik^5u - \frac{4\pi k^2}{3}D_1\Big)\hat{u} + \Big(\frac{\pi}{3}ik^3\nabla u + 2\pi\boldsymbol{E}\Big).\nabla\hat{u}\Big](\boldsymbol{x}^\circ) + \frac{1}{2}\int_{S^{\text{obs}}}\frac{\partial^2\varphi}{\partial u^2}(u(\boldsymbol{\xi}))\,V_1^2(\boldsymbol{\xi};\boldsymbol{x}^\circ)\,\mathrm{d}\Gamma_\xi \qquad (70)$$

where the scalar constant $D_1(\boldsymbol{x}^\circ)$ and the vector constant $\boldsymbol{E}(\boldsymbol{x}^\circ)$ are again defined by (40).

Expansion of the cost function for a small spherical obstacle. Summing up, the expansion of $J(\varepsilon,\boldsymbol{x}^\circ)$ up to order $O(\varepsilon^6)$ is

$$J(\varepsilon,\boldsymbol{x}^\circ) = J(0) + J^{(6)}(\varepsilon;\boldsymbol{x}^\circ) + o(\varepsilon^6) \qquad \text{with} \quad J^{(6)}(\varepsilon;\boldsymbol{x}^\circ) = A_3(\boldsymbol{x}^\circ)\varepsilon^3 + A_5(\boldsymbol{x}^\circ)\varepsilon^5 + A_6(\boldsymbol{x}^\circ)\varepsilon^6 \qquad (71)$$

where the coefficients $A_3(\boldsymbol{x}^\circ)$, $A_5(\boldsymbol{x}^\circ)$ and $A_6(\boldsymbol{x}^\circ)$ are given by (60), (69) and (70), respectively. Again, $A_3(\boldsymbol{x}^\circ) = |\mathscr{B}|\,\mathcal{T}(\boldsymbol{x}^\circ)$, where $\mathcal{T}(\boldsymbol{x}^\circ)$ is the topological derivative defined by (5).

2.5. Formulation in terms of a Green's function

An alternative approach for establishing the expansion of $J(\varepsilon;\boldsymbol{x}^\circ)$ is based on integral equations and representations formulated in terms of the Green's function associated with Ω and the boundary condition structure of the forward problem (1) rather than the the full-space fundamental solution. Such Green's function $\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi})$ satisfies

$$(\Delta_\xi + k^2)\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}) + \delta(\boldsymbol{\xi}-\boldsymbol{x}) = 0 \ \ (\text{in }\Omega), \qquad \mathcal{H}(\boldsymbol{x},\boldsymbol{\xi}) = 0 \ \ (\text{on }S_{\text{N}}), \quad \mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}) = 0 \ \ (\text{on }S_{\text{D}}) \qquad (72)$$

where $\mathcal{H}(\boldsymbol{x},\boldsymbol{\xi};\boldsymbol{n}) = \nabla\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}).\boldsymbol{n}(\boldsymbol{\xi})$ is the normal derivative of $\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi})$. On using $(\mathcal{G},\mathcal{H})$ instead of (G,H) in the governing integral equation (15) for the scattered field v^ε, the integrals over S_{N} and S_{D} vanish, which is of course the motivation behind the choice of boundary conditions in (72). The scattered field is as a result governed by the integral equation

$$\frac{1}{2}v^\varepsilon(\boldsymbol{x}) + \int_{\Gamma_\varepsilon}\mathcal{H}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{n})v^\varepsilon(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi = \int_{B_\varepsilon}\big[\nabla\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}).\nabla u(\boldsymbol{\xi}) - k^2\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{\xi})\big]\,\mathrm{d}V_\xi \quad (\boldsymbol{x}\in\Gamma_\varepsilon) \qquad (73)$$

for which only collocation points on Γ_ε are needed, and the integral representation formula

$$v^\varepsilon(\boldsymbol{x}) = -\int_{\Gamma_\varepsilon} \mathcal{H}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{n})v^\varepsilon(\boldsymbol{\xi})\,\mathrm{d}\Gamma_\xi + \int_{B_\varepsilon} [\nabla\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}).\nabla u(\boldsymbol{\xi}) - k^2\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{\xi})]\,\mathrm{d}V_\xi \quad (\boldsymbol{x}\in\Omega_\varepsilon) \tag{74}$$

valid for any $\boldsymbol{x}\in\Omega_\varepsilon$, and in particular for $\boldsymbol{x}\in S^{\text{obs}}$.

It is convenient to split $(\mathcal{G},\mathcal{H})$ according to

$$\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}) = G(\boldsymbol{x},\boldsymbol{\xi}) + \hat{G}(\boldsymbol{x},\boldsymbol{\xi})\,, \qquad \mathcal{H}(\boldsymbol{x},\boldsymbol{\xi};\boldsymbol{n}) = H(\boldsymbol{x},\boldsymbol{\xi};\boldsymbol{n}) + \hat{H}(\boldsymbol{x},\boldsymbol{\xi};\boldsymbol{n}) \tag{75}$$

where (G,H) is the singular free-space fundamental solution and the complementary part $(\hat{G},\hat{H})$ is not singular at $\boldsymbol{\xi}=\boldsymbol{x}$. On incorporating the decomposition (75) into (73), one finds that the resulting integral equation is identical to that obtained from (15) by removing the integrals over S_{N} and S_{D} and adding to the right-hand side the contribution

$$\mathcal{F}(\boldsymbol{x};\boldsymbol{x}^{\text{o}},\varepsilon) = \int_{B_\varepsilon} [\nabla\hat{G}(\boldsymbol{x},\boldsymbol{\xi}).\nabla u(\boldsymbol{\xi}) - k^2\hat{G}(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{\xi})]\,\mathrm{d}V_\xi - \int_{\Gamma_\varepsilon} \hat{G}(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{n})\,\mathrm{d}\Gamma_\xi \qquad (\boldsymbol{x}\in\Gamma_\varepsilon) \tag{76}$$

This contribution is in fact the one that gives rise to the constants $D_1(\boldsymbol{x}^{\text{o}})$, $D_2(\boldsymbol{x}^{\text{o}})$, $\boldsymbol{E}(\boldsymbol{x}^{\text{o}})$ in eqns (37) and (38). Expanding (76) in powers of ε leads to

$$\mathcal{F}(\boldsymbol{x};\boldsymbol{x}^{\text{o}},\varepsilon) = D_1(\boldsymbol{x}^{\text{o}})\varepsilon^3 + [\boldsymbol{E}(\boldsymbol{x}^{\text{o}}).\bar{\boldsymbol{x}}]\varepsilon^4 + o(\varepsilon^4) \tag{77}$$

with

$$D_1(\boldsymbol{x}^{\text{o}}) = |\mathscr{B}|\,[\nabla u(\boldsymbol{x}^{\text{o}}).(\boldsymbol{I}-\boldsymbol{\mathcal{A}}_1).\nabla\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}}) - k^2\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}})u(\boldsymbol{x}^{\text{o}})] \tag{78}$$

$$D_2(\boldsymbol{x}^{\text{o}}) = -\,|\mathscr{B}|\,[\nabla u(\boldsymbol{x}^{\text{o}}).\boldsymbol{\mathcal{A}}_1'.\nabla^{(2)}\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}}) + \nabla^{(2)}u(\boldsymbol{x}^{\text{o}}).\boldsymbol{\mathcal{A}}_2^2.\nabla\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}}) + k^2u(\boldsymbol{x}^{\text{o}})\boldsymbol{\mathcal{A}}_2^0.\nabla\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}})] \tag{79}$$

$$\boldsymbol{E}(\boldsymbol{x}^{\text{o}}) = |\mathscr{B}|\,[\nabla_x\nabla\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}}).(\boldsymbol{I}-\boldsymbol{\mathcal{A}}_1).\nabla u(\boldsymbol{x}^{\text{o}}) - k^2\nabla_x\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}})u(\boldsymbol{x}^{\text{o}})] \tag{80}$$

When spherical obstacles are considered, simplifications arise as in section 2.4, and in particular formulae (59) may be invoked. Assuming again that $\boldsymbol{x}^{\text{o}}$ lies at the center of B_ε, one obtains as a result the simpler expressions

$$D_1(\boldsymbol{x}^{\text{o}}) = \frac{4\pi}{3}[\frac{3}{2}\nabla u(\boldsymbol{x}^{\text{o}}).\nabla\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}}) - k^2\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}})u(\boldsymbol{x}^{\text{o}})] \tag{81}$$

$$D_2(\boldsymbol{x}^{\text{o}}) = 0 \tag{82}$$

$$\boldsymbol{E}(\boldsymbol{x}^{\text{o}}) = \frac{4\pi}{3}[\frac{3}{2}\nabla_x\nabla\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}}).\nabla u(\boldsymbol{x}^{\text{o}}) - k^2\nabla_x\hat{G}(\boldsymbol{x}^{\text{o}},\boldsymbol{x}^{\text{o}})u(\boldsymbol{x}^{\text{o}})] \tag{83}$$

In practice, using the Green's function defined by (72) instead of the full-space fundamental solution is mostly useful for the few simple geometrical configurations where $\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi})$ has a relatively simple expression. This includes in particular the case of a half-space with Dirichlet or Neumann conditions, where $\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi})$ is obtained from $G(\boldsymbol{x},\boldsymbol{\xi})$ by the well-known method of images.

3. PRELIMINARY SEARCH BASED ON $O(\varepsilon^6)$ EXPANSION

The functions $A_3(\boldsymbol{x}^{\text{o}}), A_4(\boldsymbol{x}^{\text{o}}), A_5(\boldsymbol{x}^{\text{o}}), A_6(\boldsymbol{x}^{\text{o}})$ can be computed on a search grid at a computational cost which is of the order of one forward solution in the reference medium (note in particular that $u, \hat{u}, V_1, V_2$ are governed by the same integral operator). Expansions of the form (48), or (71) under the assumption of a spherical small obstacle, offer the option of minimizing the approximate polynomial expression $J^{(6)}(\varepsilon,\boldsymbol{x}^{\text{o}})$. This is a simple and inexpensive task, which can be performed for locations $\boldsymbol{x}^{\text{o}}$ spanning a search grid, thereby defining a global preliminary search technique. The values of $\boldsymbol{x}^{\text{o}}$ and ε leading to an absolute minimum of $J^{(6)}(\varepsilon,\boldsymbol{x}^{\text{o}})$ over the search grid can then be subsequently used as an initial guess for a full-fledged iterative inversion algorithm. Let

$$J^{(6)}_{\text{min}}(\boldsymbol{x}^{\text{o}}) = \min_{\varepsilon} J^{(6)}(\varepsilon;\boldsymbol{x}^{\text{o}})\,, \qquad R(\boldsymbol{x}^{\text{o}}) = \arg\,\min_{\varepsilon} J^{(6)}(\varepsilon;\boldsymbol{x}^{\text{o}}) \tag{84}$$

Then, choose

$$\boldsymbol{x}^{\text{obst}} = \arg\,\min_{\boldsymbol{x}^{\text{o}}} J^{(6)}_{\text{min}}(\boldsymbol{x}^{\text{o}}) \qquad R^{\text{obst}} = R(\boldsymbol{x}^{\text{obst}})$$

as the best estimate of the obstacle location and size, where $\boldsymbol{x}^{\text{o}}$ are all points of a (finite) search grid.

This idea is now illustrated on a numerical example.

4. NUMERICAL EXAMPLE

To illustrate the proposed approach, the identification of a spherical rigid scatterer (center $\boldsymbol{x}^{\text{true}} = (2, 1.2, -3)$, radius $R^{\text{true}} = 0.5$) embedded in an acoustic medium occupying the half-space $x_3 \le 0$, and on the basis of a least-squares cost function defined by (2) with $2\varphi(w) = \overline{(w-u^{\text{obs}})}(w-u^{\text{obs}})$, is considered.

Synthetic data is created by means of a (20×20) array of point sources placed on the surface. The scattered

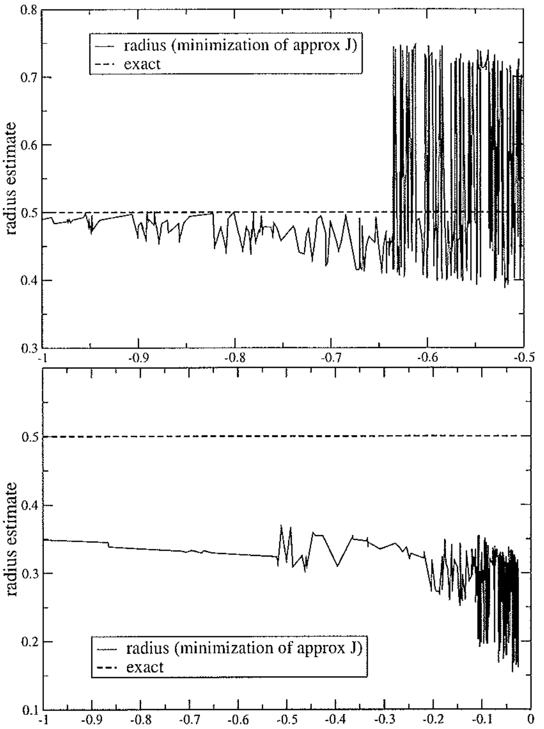

Figure 1: Radius estimates $R(\boldsymbol{x}^{\circ})$ at all grid points, plotted against the values $J^{(6)}_{\min}(\boldsymbol{x}^{\circ})$ achieved by the approximate cost function; R^{obst} corresponds to leftmost point. Cases $k=0.5$ (top) and $k=2$ (bottom).

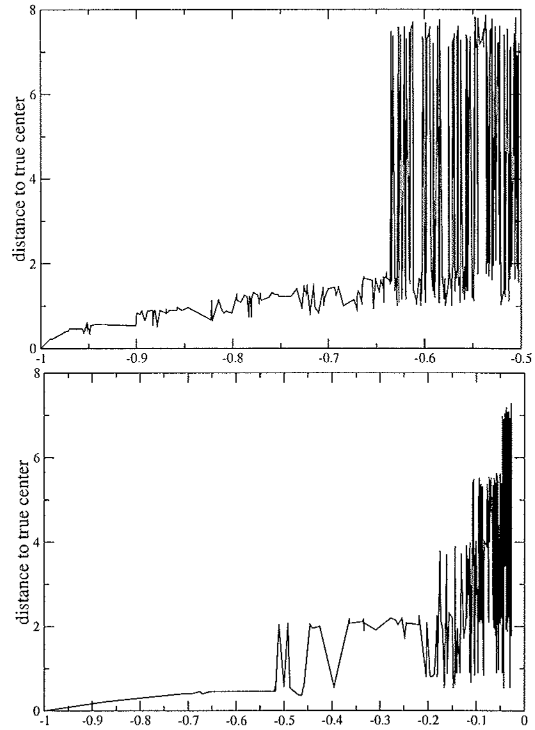

Figure 2: Distances $\|\boldsymbol{x}^{\circ}-\boldsymbol{x}^{\text{true}}\|$ for all grid points, plotted against the values $J^{(6)}_{\min}(\boldsymbol{x}^{\circ})$ achieved by the approximate cost function; leftmost point corresponds to $\|\boldsymbol{x}^{\text{obst}}-\boldsymbol{x}^{\text{true}}\|$. Cases $k=0.5$ (top) and $k=2$ (bottom).

field is then recorded on the same (20×20) array. This is a situation of limited aperture, since measurements are available only on part of the surface of the half-space. Two wavenumbers have been considered, namely $k=0.5, 2$.

The previously outlined search technique has been performed on a search grid spanning $-5\le x_1,x_2\le 5, -5\le x_3\le -1$. To investigate the efficiency of the search, the radius $R(\boldsymbol{x}^{\circ})$ found according to (84) and the distances $\|\boldsymbol{x}^{\circ}-\boldsymbol{x}^{\text{true}}\|$, obtained at each search grid point $\boldsymbol{x}^{\circ}$, are ranked in Figures 1, 2 according to the value of $J^{(6)}_{\min}(\boldsymbol{x}^{\circ})$, so that the best estimate of the obstacle location and size corresponds on each graph to the leftmost point. The correct location is found in both cases. The estimated radius R^{obst} is more accurate for the lower-frequency case $k=0.5$.

The results presented have been obtained on the basis of a MATLAB implementation, which includes the boundary element code used for computing the synthetic data u^{true} and the direct and adjoint solutions u, $\hat{u}$.

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